HILBERT SERIES OF ALGEBRAS ASSOCIATED TO DIRECTED GRAPHS

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Abstract. We compute the Hilbert series of some algebras associated to directed graphs and related to factorizations of noncommutative polynomials.

1. Introduction

In [3] we introduced a new class of algebras $A(\Gamma)$ associated to layered directed graphs Γ. These algebras arose as generalizations of the algebras Q_n (which are related to factorizations of noncommutative polynomials, see [2, 5, 9]), but the new class of algebras seems to be interesting by itself.

Various results have been proven for algebras $A(\Gamma)$. In [3] we constructed a linear basis in $A(\Gamma)$. In [7] we showed that algebras $A(\Gamma)$ are defined by quadratic relations for a large class of directed graphs and proved that in this case they are Koszul algebras. It follows immediately that the dual algebras to $A(\Gamma)$ are also Koszul and that their Hilbert series are related.

In this paper we continue to study algebras $A(\Gamma)$. In Section 2 we recall the definition of the algebra $A(\Gamma)$ and the construction of a basis for $A(\Gamma)$ given in [3]. In Section 3 we prove the main result of the paper, an expression for the Hilbert series, $H(A(\Gamma), t)$ of the algebra $A(\Gamma)$ corresponding to a layered graph Γ with a unique element $*$ of level 0. In stating this we denote the level of v by $|v|$ and write $v > w$ to indicate that v and w are vertices of the directed graph Γ and that there is a directed path from v to w. Then we have:

$$H(A(\Gamma), t) = \frac{1-t}{1 + \sum_{v_1 > v_2 \cdots > v_\ell \geq *} (-1)^\ell t^{|v_1| - |v_\ell| + 1}}.$$

The proof uses matrices $\zeta(t)$ and $\zeta(t)^{-1}$ which generalize the zeta function and the Möbius function for partially ordered sets.

In Section 4 we specialize our results to the case of the Hasse graph of the lattice of subsets of a finite set, giving a derivation of the Hilbert series for the algebras Q_n that is shorter and more conceptual than that in [2]. In Section 5 we treat the case of the Hasse graph of the lattice of subspaces of a finite-dimensional vector space over a finite field. Finally, in Section 6, we define the complete layered graph $\mathbf{C}[m_n, m_{n-1}, \ldots, m_1, m_0]$ and compute the Hilbert series of $A(\mathbf{C}[m_n, m_{n-1}, \ldots, m_1, 1])$.

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2. The algebra $A(\Gamma)$

We begin by recalling the definition of the algebra $A(\Gamma)$. Let $\Gamma = (V, E)$ be a **directed graph**. That is, V is a set (of vertices), E is a set (of edges), and $\mathbf{t} : E \to V$ and $\mathbf{h} : E \to V$ are functions. ($\mathbf{t}(e)$ is the *tail* of e and $\mathbf{h}(e)$ is the *head* of e.)

We say that Γ is **layered** if $V = \cup_{i=0}^{n} V_i$, $E = \cup_{i=1}^{n} E_i$, $\mathbf{t} : E_i \to V_i$, $\mathbf{h} : E_i \to V_{i-1}$. If $v \in V_i$ we will write $|v| = i$.

We will assume throughout the remainder of the paper that $\Gamma = (V, E)$ is a layered graph with $V = \cup_{i=0}^{n} V_i$, that $V_0 = \{*\}$, and that, for every $v \in V_+ = \cup_{i=1}^{n} V_i$, $\{e \in E \mid \mathbf{t}(e) = v\} \neq \emptyset$. For each $v \in V_+$ fix, arbitrarily, some $e_v \in E$ with $\mathbf{t}(e_v) = v$.

If $v,\ w \in V$, a **path** from v to w is a sequence of edges $\pi = \{e_1, e_2, \dots, e_m\}$ with $\mathbf{t}(e_1) = v$, $\mathbf{h}(e_m) = w$ and $\mathbf{t}(e_{i+1}) = \mathbf{h}(e_i)$ for $1 \le i < m$. We write $v = \mathbf{t}(\pi)$, $w = \mathbf{h}(\pi)$. We also write $v > w$ if there is a path from v to w. Define $P_\pi(\tau) = (\tau - e_1)(\tau - e_2)\dots(\tau - e_m) \in T(E)[\tau]$ and write

$$P_\pi(\tau) = \sum_{j=0}^{n} e(\pi, j)\tau^{m-j}.$$

Let π_v denote the path $\{e_1, \dots, e_{|v|}\}$ from v to $*$ with $e_1 = e_v, e_{i+1} = e_{\mathbf{h}(e_i)}$ for $1 \le i < |v|$, and $\mathbf{h}(e_{|v|}) = *$.

Recall that R is the ideal of $T(E)$ generated by

$$\{e(\pi_1, k) - e(\pi_2, k) \mid \mathbf{t}(\pi_1) = \mathbf{t}(\pi_2), \mathbf{h}(\pi_1) = \mathbf{h}(\pi_2),\ 1 \le k \le l(\pi_1)\}.$$

The algebra $A(\Gamma)$ is the quotient $T(E)/R$.

For $v \in V_+$ and $1 \le k \le |v|$ we define $\hat{e}(v, k)$ to be the image in $A(\Gamma)$ of the product $e_1 \dots e_k$ in $T(E)$ where $\pi_v = \{e_1, \dots, e_{|v|}\}$.

If $(v, k), (u, l) \in V \times \mathbf{N}$ we say (v, k) **covers** (u, l) if $v > u$ and $k = |v| - |u|$. In this case we write $(v, k) \gtrdot (u, l)$. (In [3] we used different terminology and notation: if $(v, l) \gtrdot (u, l)$ we said (v, l) can be composed with (u, l) and wrote $(v, l) \models (u, l)$.)

The following theorem is proved in [3, Corollary 4.5].

Theorem 1. *Let $\Gamma = (V, E)$ be a layered graph, $V = \cup_{i=0}^{n} V_i$, and $V_0 = \{*\}$ where $*$ is the unique minimal vertex of Γ. Then*

$$\{\hat{e}(v_1, k_1) \dots \hat{e}(v_\ell, k_\ell) | l \ge 0, v_1, \dots, v_\ell \in V_+, 1 \le k_i \le |v_i|, (v_i, k_i) \not\gtrdot (v_{i+1}, k_{i+1})\}$$

is a basis for $A(\Gamma)$.

3. The Hilbert series of $A(\Gamma)$

Let $h(t)$ denote the Hilbert series $H(A(\Gamma), t)$, where Γ is a layered graph with unique minimal element $*$ of level 0. If $X \subseteq A(\Gamma)$ is a set of homogeneous elements (so $X = \cup_{i=0}^{\infty} X_i$ where $X_i = X \cap A(\Gamma)_i$), denote the "graded cardinality" $\sum_{i=0}^{\infty} |X_i| t^i$ of X by $||X||$. Let B denote the basis for $A(\Gamma)$ described in Theorem 1 and, for $v \in V_+$, let $B_v = \{\hat{e}(v_1, k_1) \dots \hat{e}(v_\ell, k_\ell) \in B | v_1 = v\}$. Then $B = \{1\} \cup \bigcup_{v \in V_+} B_v$. Let $h_v(t)$ denote the graded dimension of the subspace of $A(\Gamma)$ spanned by B_v. Since B is linearly independent, we have $||B|| = h(t)$ and $||B_v|| = h_v(t)$. Then

$$||B|| = h(t) = 1 + \sum_{v \in V_+} h_v(t)$$

Let $C_v = \bigcup_{k=1}^{|v|} \hat{e}(v,k)B$. Then

$$||C_v|| = (t + \cdots + t^{|v|})h(t) = t\left(\frac{t^{|v|}-1}{t-1}\right)h(t).$$

Now $C_v \supseteq B_v$. Let D_v denote the compliment of B_v in C_v. Then

$$D_v = \{\hat{e}(v,k)\hat{e}(v_1,k_1)\ldots\hat{e}(v_\ell,k_{ell}) | 1 \le k \le |v|,$$

$$(v,k) \gtrdot (v_1,k_1), \hat{e}(v_1,k_1)\ldots\hat{e}(v_\ell,k_\ell) \in B\}$$

and so

$$D_v = \bigcup_{v>v_1>*} \hat{e}(v,|v|-|v_1|)B_{v_1}.$$

Then $||D_v|| = \sum_{v>v_1>*} t^{|v|-|v_1|}h_{v_1}(t)$ and so

$$h_v(t) = ||B_v|| = ||C_v|| - ||D_v|| = t\left(\frac{t^{|v|}-1}{t-1}\right)h(t) - \sum_{v>w>*} t^{|v|-|w|}h_w(t).$$

This equation may be written in matrix form. Arrange the elements of V in nonincreasing order and index the elements of vectors and matrices by this ordered set. Let $\mathbf{h}(t)$ denote the column vector with entry $h_v(t)$ in the v-position (where we set $h_*(t) = 1$), let $\mathbf{u}$ denote the vector with $t^{|v|}$ in the v-position, $\mathbf{e}_*$ denote the vector with δ_{*v} in the v-position, let $\mathbf{1}$ denote the column vector all of whose entries are 1, and let $\zeta(t)$ denote the matrix with entries $\zeta_{v,w}(t)$ for $v, w \in V$ where $\zeta_{v,w}(t) = t^{|v|-|w|}$ if $v \ge w$ and 0 otherwise. Note that

$$\zeta(t)\mathbf{e}_* = \mathbf{u}.$$

Then we have

$$\zeta(t)(\mathbf{h}(t) - \mathbf{e}_*) = \frac{t}{t-1}(\mathbf{u} - \mathbf{1})h(t)$$

and so

$$\mathbf{h}(t) - \mathbf{e}_* = \frac{t}{t-1}(\mathbf{u} - \zeta(t)^{-1}\mathbf{1})h(t).$$

Then

$$\mathbf{1}^T(\mathbf{h}(t) - \mathbf{e}_*) = \frac{t}{t-1}(\mathbf{1}^T\mathbf{u} - \mathbf{1}^T\zeta(t)^{-1}\mathbf{1})h(t)$$

or

$$h(t) - 1 = \frac{t}{t-1}(1 - \mathbf{1}^T\zeta(t)^{-1}\mathbf{1})h(t).$$

Consequently, we have

Lemma 1.

$$\frac{1-t}{h(t)} = 1 - t\mathbf{1}^T\zeta(t)^{-1}\mathbf{1}.$$

Now $N(t) = \zeta(t) - I$ is a strictly upper triangular matrix and so $\zeta(t)$ is invertible. In fact, $\zeta(t)^{-1} = I - N(t) + N(t)^2 - \ldots$ and so the (v,w)-entry of $\zeta(t)^{-1}$ is

$$\sum_{v=v_1>\cdots>v_l=w\ge*} (-1)^{l+1}t^{|v|-|w|}.$$

Combining this remark with Lemma 1 we obtain the following result.

Theorem 2. *Let Γ be a layered graph with unique minimal element $*$ of level 0 and $h(t)$ denote the Hilbert series of $A(\Gamma)$. Then*

$$\frac{1-t}{h(t)} = 1 + \sum_{v_1 > v_2 \cdots > v_\ell \geq *} (-1)^\ell t^{|v_1|-|v_\ell|+1}.$$

We remark that the matrices $\zeta(1)$ and $\zeta(1)^{-1}$ are well-known as the zeta-matrix and the Möbius-matrix of V (cf. [8]).

In the remaining sections of this paper we will use Theorem 2 to compute the Hilbert series of the algebras $A(\Gamma)$ associated with certain layered graphs.

4. The Hilbert series of the algebra associated with the Hasse graph of the lattice of subsets of $\{1,\ldots,n\}$

Let Γ_n denote the Hasse graph of the lattice of all subsets of $\{1,\ldots,n\}$. Thus the vertices of Γ_n are subsets of $\{1,\ldots,n\}$, the order relation $>$ is set inclusion $\supset$, the level $|v|$ of a set v is its cardinality, and the unique minimal vertex $*$ is the empty set $\emptyset$. Then the algebra $A(\Gamma_n)$ is the agebra Q_n defined in [5]. In this section we will prove the following theorem (from [2]). The present proof is much shorter and more conceptual than that in [2].

Theorem 3.

$$H(Q_n, t) = \frac{1-t}{1-t(2-t)^n}.$$

Our computations depend on the following lemma and corollary.

Lemma 2. *Let w be a finite set. Then*

$$\sum_{w \supset w_2 \supset \cdots \supset w_\ell = \emptyset} (-1)^\ell = (-1)^{|w|+1}.$$

Proof. If $|w| = 1$, both sides are $+1$. Assume the result holds for all sets of cardinality $< |w|$. Then

$$\sum_{w \supset w_2 \supset \cdots \supset w_\ell = \emptyset} (-1)^\ell = \sum_{w \supset w_2 \supseteq \emptyset} \sum_{w_2 \supset \cdots \supset w_\ell = \emptyset} (-1)^\ell$$

and, by the induction assumption, this is equal to

$$\sum_{w \supset w_2 \supseteq \emptyset} (-1)^{|w_2|}.$$

Since

$$\sum_{w \supset w_2 \supseteq \emptyset} (-1)^{|w_2|} = \sum_{w \supseteq w_2 \supseteq \emptyset} (-1)^{|w_2|} - (-1)^{|w|} = 0 + (-1)^{|w|+1} = (-1)^{|w|+1}$$

the proof is complete. $\square$

Corollary 1. *Let $v \supseteq w$ be finite sets. Then*

$$\sum_{v = v_1 \supset v_2 \supset \cdots \supset v_\ell = w} (-1)^\ell = (-1)^{|v|-|w|+1}.$$

Proof. Let w' denote the complement of w in v. Sets u satisfying $v \subseteq u \subseteq w$ are in one-to-one correspondence with subsets of w' via the map $u \mapsto u \cap w'$. Thus

$$\sum_{v=v_1 \supset v_2 \supset \cdots \supset v_\ell = w} (-1)^\ell = \sum_{w'=v'_1 \supset \cdots \supset v'_\ell = \emptyset} (-1)^\ell.$$

By the lemma, this is $(-1)^{|w'|+1}$, giving the result. $\square$

To prove the theorem we observe that

$$\sum_{\substack{v_1 \supset v_2 \supset \cdots \supset v_\ell \supseteq v_\ell \supseteq \emptyset \\ \ell \geq 1}} (-1)^\ell t^{|v_1|-|v_\ell|+1} = \sum_{\{1,\dots,n\} \supseteq v_1 \supseteq \emptyset} t^{|v_1|-|v_\ell|+1} \sum_{v_1 \supset \cdots \supset v_\ell \supseteq \emptyset} (-1)^\ell.$$

By Corollary 1, this is

$$\sum_{\{1,\dots,n\} \supseteq v_1 \supseteq v_\ell \supseteq \emptyset} t^{|v_1|-|v_\ell|+1} (-1)^{|v_1|-|v_\ell|+1}.$$

Let u denote the compliment of v_ℓ in v_1 and u' denote the complement of u in $\{1, \dots, n\}$. Then the coefficient of t^{k+1} in the above expression is the number of ways of choosing a k-element subset $u \subseteq \{1, \dots, n\}$ times the number of ways of choosing a subset $v \subseteq u'$. This is $\binom{n}{k} 2^{n-k}$. Thus

$$\sum_{\substack{v_1 \supset v_2 \supset \cdots \supset v_\ell \supseteq \emptyset \\ \ell \geq 1}} (-1)^l t^{|v_1|-|v_\ell|+1} = \sum_{k=0}^{n} \binom{n}{k} 2^{n-k} (-t)^{k+1} = -t(2-t)^k.$$

In view of Theorem 2, this completes the proof of the theorem.

5. The Hilbert series of algebras associated with the Hasse graph of the lattice of subspaces of a finite-dimensional vector space over a finite field

We will denote by $\mathbf{L(n, q)}$ the Hasse graph of the lattice of subspaces of an n-dimensional space over the field $\mathbf{F_q}$ of q elements. Thus the vertices of $\mathbf{L(n, q)}$ are subspaces of ${\mathbf{F_q}}^n$, the order relation $>$ is inclusion of subspaces $\supset$, the level $|U|$ of a subspace U is its dimension, and the unique minimal vertex $*$ is the zero subspace (0).

Theorem 4.

$$\frac{1-t}{H(A(\mathbf{L(n, q)}), t)} = 1 - t \sum_{m=0}^{n} \binom{n}{m}_q (1-t)(1-tq) \dots (1-tq^{n-m-1}).$$

Our proof depends on the following lemma and corollary.

Lemma 3. *Let U be a finite-dimensional vector space over $\mathbf{F_q}$. Then*

$$\sum_{\substack{U=U_1 \supset U_2 \supset \cdots \supset U_\ell = (0) \\ l \geq 1}} (-1)^\ell = (-1)^{|U|+1} q^{\binom{|U|}{2}}.$$

Proof. If $|U| = 0$, the sum occuring in the lemma has a single term corresponding to $l = 1, U = U_1 = (0)$. Then both sides of the expression in the lemma are equal to -1. Now let U be a finite-dimensional vector space and assume the result holds for all spaces of dimension less than $|U|$. Then

$$\sum_{\substack{U=U_1\supset U_2\supset\cdots\supset U_\ell=(0)\\ l\geq 1}} (-1)^\ell = \sum_{U=U_1\supset U_2} \sum_{\substack{U_2\supset\cdots\supset U_\ell=(0)\\ l\geq 1}} (-1)^\ell.$$

By the induction assumption, this is equal to

$$= \sum_{U\supset U_2} (-1)^{|U_2|} q^{\binom{|U_2|}{2}}.$$

It is well-known that the number of m-dimensional subspaces of the space U is given by the q-binomial coefficient $\binom{|U|}{m}_q$.

Hence

$$\sum_{\substack{U=U_1\supset U_2\supset\cdots\supset U_\ell=(0)\\ l\geq 1}} (-1)^\ell = \sum_{|U_2|=0}^{|U|-1} \binom{|U|}{|U_2|}_q (-1)^{|U_2|} q^{\binom{|U_2|}{2}}.$$

Recall the q-binomial theorem

$$\prod_{i=0}^{m-1}(1+xq^i) = \sum_{j=0}^{m} \binom{m}{j}_q q^{\binom{j}{2}} x^j.$$

Set $x=-1$. Then the $i=0$ factor in the product is 0 and so we have

$$\sum_{j=0}^{m-1} \binom{m}{j}_q (-1)^j q^{\binom{j}{2}} = (-1)^{m+1} q^{\binom{m}{2}}.$$

Thus

$$\sum_{\substack{U=U_1\supset U_2\supset\cdots\supset U_\ell=(0)\\ l\geq 1}} (-1)^\ell = (-1)^{|U|+1} q^{\binom{|U|}{2}}$$

as required. $\square$

Corollary 2. *Let $V \supseteq W$ be subspaces of $\mathbf{F}_q$. Then*

$$\sum_{V=V_1\supset V_2\supset\cdots\supseteq V_\ell=W} (-1)^\ell = (-1)^{|V/W|+1} q^{\binom{|V/W|}{2}}.$$

Proof. Since subspaces $Y, V \supseteq Y \supseteq W$, are in one-to-one correspondence with subspaces of V/W via the map $Y \mapsto Y/W$, this is immediate from the lemma. $\square$

To prove the theorem, we observe that

$$\sum_{\substack{V_1\supset V_2\supset\cdots\supset V_\ell\supseteq(0)\\ l\geq 1}} (-1)^\ell t^{|V_1/V_\ell|+1} = \sum_{\mathbf{F_q}^n\supseteq V_1\supseteq V_\ell\supseteq(0)} t^{|V_1/V_\ell|+1} \sum_{\substack{V_1\supset V_2\supset\cdots\supset V_\ell\supseteq(0)\\ \ell\geq 1}} (-1)^\ell.$$

By Corollary 2, this is equal to

$$\sum_{\mathbf{F_q}^n\supseteq V_1\supseteq V_\ell\supseteq(0)} t^{|V_1/V_\ell|+1}(-1)^{|V_1/V_\ell|+1} q^{\binom{|V_1/V_\ell|}{2}}.$$

Set $|v_\ell| = m$ and $|V_1/V_\ell| = k$. Then the number of possible V_ℓ is $\binom{n}{m}_q$ and, for fixed $V_\ell l$, the number of possible V_1 is the number of k-dimensional subspaces of $\mathbf{F_q}^n/V_\ell$ which is $\binom{n-m}{k}_q$. Thus

$$\sum_{\substack{V_1 \supset V_2 \supset \cdots \supset V_\ell \supseteq (0) \\ \ell \geq 1}} (-1)^\ell t^{|V_1/V_\ell|+1} = \sum_{\substack{0<k,m \\ k+m \leq n}} \binom{n}{m}_q \binom{n-m}{k}_q (-t)^{k+1} q^{\binom{k}{2}}$$

$$= (-t) \sum_{m=0}^{n} \binom{n}{m}_q \sum_{k=0}^{n-m} \binom{n-m}{k}_q (-t)^k q^{\binom{k}{2}}.$$

Setting $x = -t$ in the q-binomial theorem shows that

$$\sum_{k=0}^{n-m} \binom{n-m}{k}_q (-t)^k q^{\binom{k}{2}} = \prod_{i=0}^{n-m-1} (1 - tq^i).$$

Therefore

$$\sum_{\substack{V_1 \supset V_2 \supset \cdots \supset V_\ell \supseteq (0) \\ \ell \geq 1}} (-1)^\ell t^{|V_1/V_\ell|+1} = (-t) \sum_{m=0}^{n} \binom{n}{m}_q \prod_{i=0}^{n-m-1} (1 - tq^i).$$

In view of Theorem 2, the theorem is proved.

Note that setting $q = 1$ in the expression in Theorem 4 gives $1 - t(2 - t)^n$. By Theorem 3, this is $\frac{1-t}{H(Q_n,t)}$.

Recall (cf. [10]) that if A is a quadratic algebra it has a dual quadratic algebra, denoted $A^!$ and that if A is a Koszul algebra the Hilbert series of A and $A^!$ are related by

$$H(A,t)H(A^!, -t) = 1$$

Since by [7] $A(\mathbf{L}(\mathbf{n}, \mathbf{q}))$ is a Koszul algebra, we have the following

Corollary 3.

$$H(A(\mathbf{L}(\mathbf{n}, \mathbf{q}))^!, t) = 1 + \sum_{m=0}^{n-1} \binom{n}{m}_q (1 + tq) \dots (1 + tq^{n-m-1}).$$

6. The Hilbert series of algebras associated with complete layered graphs

We say that a layered graph $\Gamma = (V, E)$ with $V = \cup_{i=0}^n V_i$ is **complete** if for every $i, 1 \leq i \leq n$, and every $v \in V_1, w \in V_{i-1}$, there is a unique edge e with $\mathbf{t}(e) = v, \mathbf{h}(e) = w$. A complete layered graph is determined (up to isomorphism) by the cardinalities of the V_i. We denote the complete layered graph with $V = \cup_{i=0}^n V_i, |V_i| = m_i$ for $0 \leq i \leq n$, by $\mathbf{C}[m_n, m_{n-1}, \dots, m_1, m_0]$. Note that the graph $\mathbf{C}[m_n, m_{n-1}, \dots, m_1, 1]$ has a unique minimal vertex of level 0 and so Theorem 2 applies to $A(\mathbf{C}[m_n, m_{n-1}, \dots, m_1, 1])$. We will show:

Theorem 5.

$$\frac{1-t}{H(A(\mathbf{C}[m_n, m_{n-1}, \dots, m_1, 1], t)} =$$

$$1 - \sum_{k=0}^{n} \sum_{a=k}^{n} (-1)^k m_a (m_{a-1} - 1)(m_{a-2} - 1) \dots (m_{a-k+1} - 1) m_{a-k} t^{k+1}.$$

Proof. We first compute

$$\sum_{\substack{v_1>v_2>\cdots>v_\ell\geq\emptyset\\ \ell\geq 1}} (-1)^\ell t^{|v_1|-|v_\ell|+1}.$$

The coefficient of t^{k+1} in the sum is

$$\sum_{\substack{v_1>v_2>\cdots>v_\ell\geq *\\ \ell\geq 1,\ |v_1|-|v_\ell|=k}} (-1)^\ell = \sum_{|v_1|=k}^{n} \sum_{\substack{v_1>\cdots>v_\ell\\ |v_1|-|v_\ell|=k}} (-1)^\ell.$$

Note that the number of chains $v_1 > \cdots > v_\ell$ with $|v_i| = a_i$ for $1 \leq i \leq \ell$ is $m_{a_1}m_{a_2}\ldots m_{a_\ell}$. Then, writing $k = |v_1| - |v_\ell|$ and $a_1 = a$ we have

$$\sum_{\substack{v_1>v_2>\cdots>v_\ell\geq *\\ \ell\geq 1}} (-1)^\ell t^{|v_1|-|v_\ell|+1} = \sum_{k=0}^{n} \left(\sum_{\substack{v_1>\cdots>v_\ell\geq *\\ \ell\geq 1,\ |v_1|-|v_\ell|=k}} (-1)^\ell \right) t^{k+1}$$

$$= \sum_{k=0}^{n} \left(\sum_{\substack{a_1>\ldots a_{\ell-1}>a_1-k\geq 0\\ \ell\geq 1}} (-1)^l m_{a_1}m_{a_2}\ldots m_{a_{\ell-1}}m_{a_1-k} \right) t^{k+1}$$

$$= \sum_{k=0}^{n} \left(\sum_{a=k}^{n} m_a(1-m_{a-1})\ldots(1-m_{a-k+1})m_{a-k} \right) t^{k+1}.$$

The theorem now follows from Theorem 2. $\square$

This result applies, in particular, to the case $m_0 = m_1 = ... = m_n = 1$. The resulting algebra $A(\mathbf{C}[1,\ldots,1])$ has n generators and no relations. Theorem 5 shows that

$$\frac{1-t}{H(A(\mathbf{C}[1,\ldots,1]),t)} = 1 - \sum_{a=0}^{n} t + \sum_{a=1}^{n} t^2 = (1-t)(1-nt).$$

Thus $H(A(\mathbf{C}[1,\ldots,1]),t) = \frac{1}{1-nt}$ and we have recovered the well-known expression for the Hilbert series of the free associative algebra on n generators.

Since by [7] the algebras associated to complete directed graphs are Koszul algebras, we have the following

Corollary 4.

$$H(A(\mathbf{C}[m_n, m_{n-1},\ldots,m_1,1])^!,t) =$$

$$1 + \sum_{k=1}^{n}\sum_{a=k}^{n} m_a(m_{a-1}-1)(m_{a-2}-1)\ldots(m_{a-k+1}-1)\, t^k.$$

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